B.Sc – I

Subject : Mathematics

Paper : Real Analysis

Semester : II

Unit – I

Short Answer Questions

1. Define Bounded below, Bounded Above sequence, Convergent sequence, Divergent sequence.

2. Find supremum, infimum, greatest member, least member of the set (i) $S = \{ \frac{(-1)^n}{n} / n \in N \}$ (ii) $S = \{ 1 + \frac{(-1)^n}{n} / n \in N \}$
(iii) $S = \{ \frac{1}{n} / n \in N \}$

3. For all Real numbers x, y show that (i) $|x+y| \leq |x| + |y|$
(ii) $|x-y| \geq \big| |x| - |y| \big|$

4. Define neighbourhood of a point, limit point of a set.

5. State and Prove Bolzano's weierstrass theorem.

6. Define convergent sequence, Bounded sequence.

7. Prove that every convergent sequence is bounded.

8. Show that (i) $\lim_{n \to \infty} \sqrt[n]{a} = 1$, If $a > 0$ (ii) $\lim_{n \to \infty} \sqrt[n]{n} = 1$

9. Show that the sequence $\{ r^n \}$ converges iff $-1 < r \leq 1$

10. Show that $\lim_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$.

11. Define cauchy sequence and also show that the sequence $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not convergent.

12. Show that $\lim_{n \to \infty} \frac{1+3+5+\cdots+(2n-1)}{n^2} = 1$

13. State and Prove Sandwich theorem.

14. Show that $\lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right] = 1$

15. Define monotonic sequence and subsequence.

Long Answer Questions

16. A necessary and sufficient condition for convergence of a sequence $\{S_n\}$ for each $\varepsilon > 0$ then there exist a +ve integer 'm' such that $|S_{n+p} - S_n| < \varepsilon, \forall\, n \geq m$ and $p \geq 1$.

17. State and prove cauchy first theorem on limits.

18. State and Prove cauchy second theorem on limits.

19. State and Prove cesaro's theorem.

20. Show that the sequences $\{a_n^{\frac{1}{n}}\}$, $\{b_n^{\frac{1}{n}}\}$ are convergent. and also find their limits. where (i) $a_n = \frac{3n!}{(n!)^3}$ (ii) $b_n = \frac{n^n}{(n+1)(n+2)\cdots(n+n)}$

21. S.T A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

22. Show that the sequence $S_n = \left(1+\frac{1}{n}\right)^n$ is convergent and $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n$ lies between 2 and 3.

23. Show that sequence $\{s_n\}$ where $s_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is convergent.

24. Show that the sequence $\{s_n\}$ defined by the recursion formula $s_{n+1} = \sqrt{3 s_n}$, $s_1 = 1$ is converges to 3.

25. Show that the sequence $\{s_n\}$ defined by the recursion formula $s_{n+1} = 7 + s_n$ where $s_n = \sqrt{7}$ converges to the positive root of $x^2 - x - 7 = 0$.

26. If $\{s_n\}$ is a sequence such that $s_{n+1} = \sqrt{\frac{ab^2 + s_n^2}{a+1}}$, $b > a$ $\forall n \geq 1$ and $s_1 = a > 0$ then show that the sequence $\{s_n\}$ is an Increasing bounded above sequence and $\lim_{n \to \infty} s_n = b$.

Unit-II

SAQ

1. State and Prove Pringsheim's theorem.
2. S.T the series $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ is convergent.
3. S.T the series $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\cdots$ is convergent.
4. Test for convergence of the series whose n^{th} term is $\left[(n^3+1)^{1/3}-n\right]$
5. S.T the series $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots$ converges for $p>0$.
6. S.T the series $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\cdots$ is convergent.
7. S.T the series $\sum \sqrt{n^4+1}-\sqrt{n^4-1}$ is convergent.
8. S.T the series $\frac{1}{(\log 2)^p}+\frac{1}{(\log 3)^p}+\cdots+\frac{1}{(\log n)^p}+\cdots$ divergent for $p>0$.
9. Find left hand and Right hand limits of a function defined as $f(x)=\begin{cases}\frac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4.\end{cases}$
10. Evaluate $\lim_{x\to 0}\frac{e^{1/x}}{1+e^{1/x}}$
11. S.T $\lim_{x\to 0}\frac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-1/x}}$ does not exist.
12. Refer All problems on limits & continuity

LAQ

1. S.T. a Positive term series $\sum \frac{1}{n^p}$ is convergent if and only if $P>1$.

2. State and Prove Comparison Test First type.
3. State and Prove Cauchy's Root Test.
4. State and Prove D'Alembert's Ratio Test.
5. State and Prove Cauchy's Integral Test.
6. State and Prove Leibnitz Test.
7. If a function 'f' is continuous on a closed interval $[a,b]$ then it attains its bounds atleast once in $[a,b]$.

8. State and Prove Intermediate value theorem.
9. State and Prove Fixed Point theorem.
10. Investigate the Continuity of a function at the Point is given

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}, & \text{If } x \neq 0 \\ 1, & \text{If } x = 0 \end{cases}$$

at $x=0$.

11. Examine the Continuity at $x=1$.

$$f(x) = \begin{cases} 2x, & \text{when } 0 \leq x < 1 \\ 3, & \text{when } x = 1 \\ 4x, & \text{when } 1 < x \leq 2. \end{cases}$$

12. A function 'f' defined on R by $f(x) = \begin{cases} -x^2, & \text{If } x \leq 0 \\ 5x-4, & \text{If } 0 < x \leq 1 \\ 4x^2-3x, & \text{If } 1 < x < 2 \\ 3x+4, & \text{If } x \geq 2 \end{cases}$

Examine 'f' for continuity at $x=0,1,2$ and also discuss the Kind of discontinuity.

Unit-III

SAQ.

1. S.T the function $f(x)=x^2$ is derivable on $[0,1]$
2. S.T A function which is derivable at a point is necessarly continuous at that point.
3. Discuss the derivability of a function at $x=1$.

$$f(x) = x, \text{ If } 0 \le x < 1$$
$$= 1, \text{ If } x \geq 1.$$

4. A function 'f' defined as $f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x=0 \end{cases}$ is derivable at $x=0$ but $\lim\limits_{x \to 0} f'(x) \neq f'(0)$
5. Find $Lf'(0)$ and $Rf'(0)$ of $f(x) = \begin{cases} x(e^{\frac{1}{x}} - e^{\frac{-1}{x}}), & \text{If } x \neq 0 \\ 0, & \text{If } x=0 \end{cases}$
6. S.T the function $f(x) = |x| + |x-1|$ is derivable at all points except 0 and 1.
7. Discuss the derivability of $f(x) = \begin{cases} 2x-3, & \text{If } 0 \le x \le 2 \\ x^2-3, & \text{If } 2 < x \le 4 \end{cases}$ at $x = 2, 4$.
8. Examine the validity of the hypothesis and conclusion of Rolle's theorem.

(i) $f(x) = x^3 - 4x$ on $[-2, 2]$

(ii) $f(x) = (x-a)^m (x-b)^n$ on $[a,b]$

(iii) $f(x) = 1 - (x-1)^{2/3}$ on $[0,2]$

(iv) $f(x) = 1 - |x-1|$ on $[0,2]$

9. verify Lagrange's theorem for the following functions.

(i) $f(x) = 2x^2 - 7x + 10$ on $[2,5]$ (ii) $f(x) = \log x$ on $[\frac{1}{2}, 2]$

(iii) $f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$ (iv) $f(x) = x^{\frac{1}{3}}$ on $[-1,1]$

10. S.T $\log(1+x)$ lies between $x - \frac{x^2}{2}$ and $x - \frac{x^2}{2(1+x)}$, $\forall x > 0$

SAQ(OR)LAQ

11. S.T $\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$, If $0 < u < v$ and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

12. Expand the following functions by using Maclaurin's theorem.

(i) $\sin x$ (ii) e^x (iii) $\cos x$ (iv) $\log(1+x)$

LAQ

1. State and Prove Intermediate value theorem for derivatives.
2. State and Prove Rolle's theorem.
3. State and Prove Lagrange's mean value theorem.
4. State and Prove Cauchy's mean value theorem.
5. State and Prove Taylor's theorem.

Unit-IV (Riemann Integral)

SAQ

1. S.T a constant function K is integrable and $\int_a^b K\,dx = K(b-a)$.

2. S.T. x^2 is integrable on any interval $[0,K]$

3. S.T $(3x+1)$ is integrable on $[1,2]$ and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$.

4. S.T $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$

5. S.T. If 'f' is Integrable on $[a,b]$ then f^2 is also integrable on $[a,b]$

6. S.T $\int_0^1 f\,dx = \frac{2}{3}$ where f is Integrable.

7. S.T $\int_0^3 [x]\,dx = 3$. where $[x]$ = Greatest integer not greater than 'x'.

LAQ

1. S.T a bounded function 'f' is integrable on [a,b] iff for every $\epsilon > 0$ there exist a partition P of [a,b] such that $U(P,f) - L(P,f) < \epsilon$.

2. If f_1 and f_2 are two bounded and integrable functions on [a,b] then $f = f_1 + f_2$ also integrable on [a,b] and

$$\int_a^b f_1\,dx + \int_a^b f_2\,dx = \int_a^b f\,dx.$$

3 S.T every continuous function is Integrable.

4. S.T every monotonic function is Integrable on [a,b]

5. If a function 'f' is bounded and Integrable on [a,b] then the function $\underline{F}$ is defined as $F(x) = \int_a^x f(t)\,dt$ is continuous on [a,b] and furthermore, If 'f' is continuous at a point 'c' of [a,b] then $\underline{F}$ is derivable at 'c' and $F'(c) = f(c)$.

Imp Property

6. State and prove fundamental theorem of calculus.